POLYNOMIAL SYSTEMS ADMITTING A SIMULTANEOUS SOLUTION

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Abstract. We provide a description of a complete set of generators for the ideal that serves as the resultant ideal for n univariate polynomials of degree d. Our generators arise as maximal minors of a set of cascading matrices formed from the coefficients of the polynomials, generalising the classical Sylvester resultant of two polynomials.

1. Introduction

Fix integers $n > 1$ and $d > 1$. Consider a system

$$f_i(x) := a_{i,0}x^d + a_{i,1}x^{d-1} + \cdots + a_{i,d} = 0 \qquad 1 \leq i \leq n \tag{1}$$

of n univariate polynomials of degree d in a variable x over an algebraically closed field K. A natural question arises: *when do the polynomials f_i have a common root?*

By eliminating the variable x from the ideal $\langle f_i(x)\rangle$, we obtain a radical ideal

$$I_{d,n} \lhd K[a_{i,j}]_{1\leq i\leq n, 0\leq j\leq d}$$

in the polynomial ring of coefficients, which serves as a resultant for the set of polynomials $\{f_i(x)\}$ in the following sense.

(a) If the polynomials $f_i(x)$ have a common root, then the coefficients $a_{i,j}$ belong to the variety $V(I_{d,n})$.
(b) If the coefficients $a_{i,j}$ belong to the variety $V(I_{d,n})$, then either the polynomials $f_i(x)$ have a common root, or for all i we have $a_{i,0} = 0$.

The conclusion in (b) simply means that the associated binary forms

$$g_i(x, y) := a_{i,0}x^d + a_{i,1}x^{d-1}y + \cdots + a_{i,d}y^d$$

have a common root in $\mathbb{P}^1_{[x:y]}$. This is a Zariski closed condition in the projective space defined by the coefficients $a_{i,j}$, and is the closure of the condition that the polynomials $f_i(x)$ have a common root in $\mathbb{A}^1_x$. Thus the ideal $I_{d,n}$ is the fundamental object providing the answer to our basic question; we will call it the *resultant ideal* of the polynomial system (1), following the terminology in [5].

In this paper, we give a description of a complete set of generators of the ideal $I_{d,n}$. Aspects of this very natural and classical problem have been investigated since the 19th century. The best known is the case $n = 2$ of two equations. As observed by Sylvester, the ideal $I_{d,2}$ is principal, generated by the resultant polynomial $\mathrm{Res}(f_1, f_2) \in K[a_{i,j}]$, the determinant of a matrix known nowadays as the Sylvester matrix. For the next case $d = 2, n = 3$, the ideal $I_{2,3}$ is easily computed (at least by computer algebra), and was studied earlier in [1, Ex. 5.6, Ex. 6.6]. For small, fixed

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$n > 2$ and d, one can still give an explicit set of generators for $I_{d,n}$ via elimination. However, this quickly becomes impossible, and the answer intractable.

The ideal $I_{d,n}$, as well as the variety $V(I_{d,n})$, the locus of forms that have a common root, have also been studied from a theoretical point of view. A classical reference is van der Waerden [11, §130], where some of the properties of $I_{d,n}$ are described. However, in all modern editions, the short arguments involving this ideal are nonconstructive, only appearing as corollaries of the Nullstellensatz; older versions made more explicit use of the theory of resultants.

A question closely related to ours was already answered a long time ago by Orsinger [9, Satz 7], though this does not appear to be generally known [10], even for the case of quadratic polynomials. This is the set-theoretic question of giving polynomial conditions for the coefficients a_{ij} that guarantee the existence of a common root. Orsinger's result was rediscovered by Kakié [6] and also in greater generality by Jouanolou [5, Section 3.3.7]. The question of finding the minimal number of polynomial conditions ensuring a common root was investigated by Lyubeznik [7]. However, these results do not approach the problem in an ideal-theoretic sense: the polynomials they provide do not generate the resultant ideal $I_{d,n}$, for simple degree reasons. What they generate instead is a non-radical ideal with radical $I_{d,n}$. This phenomenon already occurs for quadratic polynomials, where for $n > 2$ a natural set of generators for $I_{2,n}$ include classical resultant quartics, some further degree-4 relations already contained in [6, 9], as well as cubic relations. Jouanolou [5] discusses many further properties of the ideal $I_{d,n}$. However, to our knowledge, the explicit description of a generating set for the ideal $I_{d,n}$ was not known before our work. In particular, as observed by Jan Stevens in private communication, our results imply that the determinantal equations considered by Orsinger and Kakié define the correct projective scheme, although they do not generate the correct ideal. See Remark 5, Corollary 15 and Proposition 16 for further discussion.

In this article we provide a description of the resultant ideal $I_{d,n}$ in the following sense: we provide

(a) a list of generators, in determinantal form, for $I_{d,n}$;
(b) a Gröbner basis for $I_{d,n}$;
(c) the degree and the dimension of the variety $X_{d,n} := V(I_{d,n}) \subset \mathbb{P}^{n(d+1)-1}$.

Here (c) is straightforward, using a natural resolution of singularities of $X_{d,n}$ via vector bundles, the subject of our Section 2. At the start of Section 3, we provide a set of determinantal elements in the resultant ideal. Our strategy to solve (a)-(b), and in particular to prove our main result Theorem 4, is as follows. First, we will pick a term order on the polynomial ring of coefficients, and a subset $G \subset I_{d,n}$, which will eventually be shown to be a Gröbner basis. We will show that the leading terms of G are square-free and that the variety defined by the corresponding initial ideal is the union of $\deg X_{d,n}$ coordinate subspaces and of dimension equal to $\dim X_{d,n}$. As we will argue, these facts establish that G is a Gröbner basis of $I_{d,n}$. We conclude the paper in Section 4 with final remarks, in particular recovering the set-theoretic description as a special case.

In Section 2, we will be working in a more general setting, where the polynomials in the system (1) can have different degrees. However, from Section 3, we focus on the case of polynomials of equal degree d. We hope to return to the more general case in later work.

We will be assuming familiarity with ideals, Gröbner bases, term orders, elimination theory and the Nullstellensatz, as presented in [8, Chapters 1–4 and 6]. We also rely on basic intersection theory, referring the interested reader to [2].

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2. Dimension and degree

We start by computing the dimension and degree of the projective variety

$$X_{d,n} = V(I_{d,n}) \subset \mathbb{P}^{n(d+1)-1},$$

the vanishing locus of the resultant ideal, defined in the Introduction. Note that different proofs of these results were given in [1, Lem. 6.3, Prop. 6.5].

Let us slightly generalize the setting: consider n bivariate, homogeneous forms

$$g_i(x,y) := a_{i,0}x^{d_i} + a_{i,1}x^{d_i-1}y + \cdots + a_{i,d_i}y^{d_i}, \quad i = 1, \ldots, n$$

of possibly distinct degrees $d_1, \ldots, d_n$. Denoting $D := \sum_{i=1}^n d_i$, the space of such forms is parameterized by the affine space K^{n+D} of coefficients $a_{i,j}$. Let $X_{d_1,\ldots,d_n} \subset \mathbb{P}^{n-1+D}$ be the locus inside the projective space of coefficients corresponding to those n-tuples of forms that have a common root in $\mathbb{P}^1$.

Proposition 1. *The set $X_{d_1,\ldots,d_n} \subset \mathbb{P}^{n-1+D}$ is an irreducible projective variety of dimension D and degree D.*

Proof. Consider the projective line $\mathbb{P}^1$ with coordinates x, y. Inside $\mathbb{P}^1 \times \mathbb{P}^{n-1+D}$, each binary form $g_i(x,y)$ defines a codimension one subvariety B_i. Projecting B_i to $\mathbb{P}^1$ makes B_i into a projective bundle of rank $n-2+D$ over $\mathbb{P}^1$, a codimension one subbundle of the trivial bundle.

We now prove that all B_i's intersect transversally. Indeed, as bundles are locally trivial, they intersect transversally if and only if they intersect transversally on every fiber. However, for fixed $[x:y]$ each g_i becomes a linear equation in a *distinct* set of variables. In particular, the linear equations are independent and thus the intersection is transversal. It follows that the variety $Y_{d_1,\ldots,d_n} := \bigcap_{i=1}^n B_i$ is also a projective bundle over $\mathbb{P}^1$ of rank $(n-1+D)-n = D-1$. Hence, $\dim Y_{d_1,\ldots,d_n} = D$.

Consider the projection $\mathbb{P}^1 \times \mathbb{P}^{n-1+D} \to \mathbb{P}^{n-1+D}$. We claim that the image of $Y_{d_1,\ldots,d_n}$ is precisely $X_{d_1,\ldots,d_n}$. Indeed, a point $([x:y],[a_{i,j}])$ belongs to $Y_{d_1,\ldots,d_n}$ if and only if $[x:y]$ is a common root of the binary forms $g_i(x,y)$. In particular, $X_{d_1,\ldots,d_n}$ is an irreducible variety. Further, the resulting map

$$\pi\colon Y_{d_1,\ldots,d_n} \to X_{d_1,\ldots,d_n}$$

is birational [5, Prop.3.3.1]. Indeed, the general fiber is a singleton, as, for general $g_i(x,y)$ having a common root, this root is unique. Thus, $\dim X_{d_1,\ldots,d_n} = \dim Y_{d_1,\ldots,d_n} = D$.

As a side remark, we note that π is not an isomorphism, as some systems have several common solutions. In particular, $X_{d_1,\ldots,d_n}$ in general is singular, while $Y_{d_1,\ldots,d_n}$ is always smooth, with π a resolution of singularities of $X_{d_1,\ldots,d_n}$.

Recall that the degree of $X_{d_1,\ldots,d_n}$ is the number of points we obtain after intersecting it with D general hyperplanes in $\mathbb{P}^{n-1+D}$. Pulling back hyperplanes of $\mathbb{P}^{n-1+D}$ by the projection map, we obtain divisors on $\mathbb{P}^1 \times \mathbb{P}^{n-1+D}$ that belong to a base-point-free linear system $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^{n-1+D}}(H_2)$, the pullback of the hyperplane system on the second factor. Intersecting $Y_{d,n}$ with D general divisors from this linear system, by Bertini's theorem we obtain a finite number k of reduced points of Y that are general in the sense that they belong to the open complement $Y_{d_1,\ldots,d_n} \setminus \mathrm{Exc}(\pi)$ of the exceptional locus of π. Let us note that as the intersection points belong to the locus where $Y_{d_1,\ldots,d_n}$ and $X_{d_1,\ldots,d_n}$ are isomorphic, to know that we obtain reduced points, it is enough to apply Bertini's theorem for the complete linear system of hyperplanes in the projective space, which holds in arbitrary characteristic of the field. It follows that $k = \deg X_{d_1,\ldots,d_n}$.

It remains to compute the number of points we obtain by intersecting $Y_{d_1,\ldots,d_n}$ with D divisors of the linear system $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^{n-1+D}}(H_2)$. Recall that the Chow ring of $\mathbb{P}^1 \times \mathbb{P}^{n-1+D}$ is $R = \mathbb{Z}[H_1, H_2]/(H_1^2, H_2^{n+D})$, with H_i the hyperplane class pulled back from each factor, the class of the point being the top nonzero intersection $H_1 H_2^{n-1+D}$.

Each divisor B_i is of degree d_i in x, y and degree 1 in the coefficient variables $a_{i,j}$. Its class is thus $d_i H_1 + H_2 \in R$. As we proved that $Y_{d_1,\ldots,d_n}$ is a transversal intersection of the hypersurfaces B_i, its class is the product $\prod_{i=1}^n (d_i H_1 + H_2) \in R$. It remains to compute the intersection with D divisors of class H_2, which are general, hence transversal by Bertini's theorem, to deduce

$$H_2^D \prod_{i=1}^n (d_i H_1 + H_2) = D \cdot H_1 H_2^{n-1+D} \in R,$$

and thus $k = D$. □

Corollary 2. *The projective variety $X_{d,n} \subset \mathbb{P}^{n(d+1)-1}$ is irreducible of dimension $\dim X_{d,n} = nd$ and degree $\deg X_{d,n} = nd$.*

Remark 3. As argued above, $Y_{d,n} \subset \mathbb{P}^{n(d+1)-1} \times \mathbb{P}^1$ is a complete intersection, and hence its ideal can be resolved by the Koszul complex [5, (1.7)]. Pushing forward this resolution along the map π, together with a standard computation[1], shows that $X_{d,n}$ is not normal. A full resolution of the ideal of the embedding $X_{d,n} \subset \mathbb{P}^{n(d+1)-1}$ is studied in [5, Section 4]. In that setting, a certain module $\tilde{\Gamma}$ surjects onto the ideal of $X_{d,n}$. Thus knowing the generators of $\tilde{\Gamma}$ would provide generators of the ideal of $X_{d,n}$. The module $\tilde{\Gamma}$ may be realized as a kernel of a map in a certain degree in the dual of a Koszul complex (see [5, Remarque 4.5.3]). As Jouanolou writes, thanks to the results of Hermann and Hilbert, this allows one in principle to obtain information about the generators of $\tilde{\Gamma}$ and hence about the generators of the ideal of $X_{d,n}$. On the other hand, computing a set of generators of this kernel is a very hard task, although algorithmically doable. The relationship between Jouanolou's resolution and our main result Theorem 4 below deserves further study.

[1]We would like to thank Jerzy Weyman for explaining this.

3. Determinantal equations and the main result

By definition, $I_{d,n}$ is the radical ideal defining the irreducible variety $X_{d,n} \subset \mathbb{P}^{n(d+1)-1}$, and hence prime (see also [5, 3.3.7]). Our next step is to construct determinantal equations in $I_{d,n}$. For $1 \leq k \leq d$, define the $nk \times (d+k)$ matrix

$$M_k = \begin{bmatrix} a_{1,0} & \cdots & a_{1,d} & & & \\ & \vdots & & & & \\ a_{n,0} & \cdots & a_{n,d} & & & \\ & a_{1,0} & \cdots & a_{1,d} & & \\ & & \vdots & & & \\ & a_{n,0} & \cdots & a_{n,d} & & \\ & & & \ddots & & \\ & & & a_{1,0} & \cdots & a_{1,d} \\ & & & \vdots & & \\ & & & a_{n,0} & \cdots & a_{n,d} \end{bmatrix},$$

where each rectangular matrix is shifted to the right by one step corresponding to the rectangle above it, there are a total of k rectangles, and the unspecified entries are all zero. If x is a common solution to the system (1), then $[x^{d+k-1}, x^{d+k-2}, \ldots, 1]^t$ lies in the kernel of M_k, hence all $(d+k) \times (d+k)$ minors of M_k lie in $I_{d,n}$. Alternatively, writing P_s for the vector space of univariate polynomials of degree at most s, M_k^t can be interpreted as the map $(P_k)^{\times n} \to P_{d+k}$, $(h_1, \ldots, h_n) \mapsto f_1 h_1 + \cdots + f_n h_n$. This map is rank deficient when x is a common root of the polynomials f_i, as then it is a common root of the entire image. Thus, the Fitting ideal corresponding to rank $(d+k)$ is contained in $I_{d,n}$.

The following is our main result.

Theorem 4. *The resultant ideal* $I_{d,n} \lhd K[a_{i,j}]_{1 \leq i \leq n, 0 \leq j \leq d}$ *is generated by all* $(d+k) \times (d+k)$ *minors of* M_k *for* $1 \leq k \leq d$.

Remark 5. In the above theorem, if n is small with respect to d, there may be no appropriately sized minors of M_k for small k. For instance, the theorem asserts that in the case $n = 2$, $I_{d,2}$ is generated by a single $2d \times 2d$ determinant, the classical Sylvester resultant $\mathrm{Res}(f_1, f_2)$.

More generally, it is easy to see that for the largest value $k = d$, some of the $2d \times 2d$ minors of M_d are just the pairwise resultants $\mathrm{Res}(f_i, f_j)$ of the original polynomials. The vanishing of the entire set of $2d \times 2d$ minors of M_d is precisely the set-theoretic condition for the existence of a common root found by Kakié [6]; we will recover this result below in Corollary 15. These degree $2d$ polynomials alone clearly cannot generate the ideal $I_{d,n}$, as the minors of the smaller M_k are of lower degree $d + k$.

At the other extreme $k = 1$, the condition on the rank of M_1 is simply the condition that if $d+1$ univariate polynomials of degree d share a root, then these polynomials are linearly dependent; this is easy to see directly. However, it is also immediate (for example for dimension reasons) that the set of equations $\mathrm{rk}\, M_1 < d+1$ is not sufficient, even set-theoretically, to force a common root.

We will use Gröbner basis techniques to prove Theorem 4. The first step is to establish a term order on the polynomial ring $K[a_{i,j}]_{1\leq i\leq n, 0\leq j\leq d}$, which is achieved in the next proposition.

Proposition 6. *There is a term order on* $K[a_{i,j}]_{1\leq i\leq n, 0\leq j\leq d}$ *with the property that the leading monomial of any minor of* M_k *is the product of its diagonal elements.*

Remark 7. Observe that the content of this proposition is sensitive to the order of the rows of M_k, even as the set of minors up to sign is not. If the order of the rows of M_k is permuted, the meaning of the diagonals of minors will change and the claim may no longer hold.

Remark 8. The proposition implies that a minor of M_k is not identically zero exactly when its diagonal contains no zeros. The fact that a zero on the diagonal implies that the minor is zero can also be seen directly.

Proof of Proposition 6. Fix an increasing sequence of positive numbers

$$x_{n,1} < x_{n-1,1} < \dots < x_{1,1} < x_{n,2} < x_{n-1,2} < \dots < x_{1,2} < x_{n,3} < \dots$$
$$\dots < x_{n,d} < x_{n-1,d} < \dots < x_{1,d}.$$

Assign weight one to each $a_{k,d}$ for $k = 1,\dots,n$. Inductively, from $l = d$ and going down to $l = 0$, assign weight $w_{k,l}$ to each $a_{k,l}$ so that $w_{k,l} - w_{k,l+1} = x_{k,l+1}$. We claim that any term order compatible with the given weights will choose the diagonal as a leading term for any minor.

For contradiction assume this is not the case and fix a minor for which the leading term is not the diagonal. Let $X = (y_{i,j})$ be the corresponding submatrix. If the leading term does not correspond to the diagonal then it must be divisible by the product $y_{i,j}y_{p,q}$ so that $i < p$ and $q < j$. We claim that replacing this term by $y_{i,q}y_{p,j}$ would increase the weight, which gives the contradiction.

Say $y_{i,j} = a_{i',j'}$ and $y_{p,q} = a_{p',q'}$. Then there is a c such that $y_{i,q} = a_{i',j'-c}$ and $y_{p,j} = a_{p',q'+c}$. In particular, we note that these are nonzero, as $q' < q'+c, j'-c \leq j'$. It remains to observe that the difference of weights $w_{i',j'-c} - w_{i',j'}$ is greater than the difference of weights $w_{p',q'} - w_{p',q'+c}$, which follows from the choice of $x_{i,j}$'s. □

Choosing a $(d+k)\times(d+k)$ minor of M_k is the same as choosing a subset of the rows of size $d+k$. Rows of M_k are naturally indexed by pairs (i,j), where $1 \leq i \leq k$ and $1 \leq j \leq n$, so we may identify such minors with their lexicographically ordered list of pairs $((i_1,j_1),\dots,(i_{d+k},j_{d+k}))$. Here, the ordering corresponds to taking rows of M_k from top to bottom.

Corollary 9. *Write* $a_{i,j} = 0$ *if* $j < 0$ *or* $j > d$*. The leading monomial of the minor* $((i_1,j_1),\dots,(i_{d+k},j_{d+k}))$ *is* $\prod_{s=1}^{d+k} a_{j_s,s-i_s}$.

For fixed minor, for each s, write $(u_s,v_s) := (j_s, s-i_s)$. When $s > 1$, we have that either $v_s \leq v_{s-1}$ or ($v_s = v_{s-1}+1$ and $u_s > u_{s-1}$). These correspond to the cases that $i_s > i_{s-1}$ and $i_s = i_{s-1}$, respectively. Restrict attention now to nonzero minors. These are exactly those for which each (u_s,v_s) is contained within the $n\times(d+1)$ lattice. We always have $v_1 = 1-i_1 \leq 0$ and $v_{d+k} = d+k-i_{d+k} \geq d$, so for a nonzero minor in particular equality holds for both. Call a walk $(u_1,v_1),\dots,(u_{d+k},v_{d+k})$ through the lattice satisfying the conditions

(1) $v_{s+1} \leq v_s$ or both $v_{s+1} = v_s + 1$ and $u_{s+1} > u_s$;
(2) $v_1 = 0$ and $v_{d+k} = d$

a *minor walk*. Every minor walk arises from an actual minor: the only thing to check is that i_s satisfies $1 \leq i_s \leq k$. These are the conditions $s - k \leq v_s \leq s - 1$. The upper bound follows from $v_1 = 0$ and $v_{s+1} \leq v_s + 1$ and the lower bound from the fact that $v_{d+k} = d$ and $v_{s-1} \geq v_s - 1$. We have shown

Proposition 10. *The nonzero $(d + k) \times (d + k)$ minors of M_k correspond exactly to minor walks of length $d + k$. The leading monomial of the minor corresponding to a walk is obtained by multiplying the variables corresponding to the visited lattice points, counted with multiplicity.*

If any subset of the minors of Theorem 4 forms a Gröbner basis, then so must a subset whose leading terms divide the leading terms of any minor. Let us construct a minimal such subset, which we will see actually corresponds to a unique set of minors. Let a *reduced* minor walk denote a minor walk which is minimal under inclusion, i.e., one for which no vertex can be deleted and remain a minor walk.

Lemma 11. *A minor walk $(u_1, v_1), \ldots, (u_{d+k}, v_{d+k})$ is reduced if and only if*

(1) $v_{s+1} \geq v_s$*;*
(2) $v_2 = 1$ *and* $v_{d+k-1} = d - 1$*;*
(3) *if* $v_{s+1} = v_s$*, then* $v_s = v_{s-1} + 1$*,* $v_{s+2} = v_{s+1} + 1$*,* $u_s \geq u_{s+2}$ *and* $u_{s-1} \geq u_{s+1}$*. In particular,* $u_s > u_{s+1}$*.*

Furthermore, a reduced minor walk visits each vertex at most once, is determined by the set of vertices it visits, and no minor walk can visit a proper subset of the visited vertices.

Proof. Let $(u_1, v_1), \ldots, (u_{d+k}, v_{d+k})$ be a minor walk. We first show if any of the conditions of the claim are violated, this walk is not reduced.

(1) If $v_{s+1} < v_s$, then $v_{s+1} \leq v_{s-1}$ and thus we can remove the s-th step.
(2) If $v_2 \neq 1$, then $v_2 = 0$ and we may remove the first step. Analogously if $v_{d+k-1} \neq d - 1$ we may remove the last step.
(3) Suppose $v_{s+1} = v_s$. If $v_s \neq v_{s-1} + 1$ or $u_{s-1} < u_{s+1}$, we may remove the s-th step. If $v_{s+2} \neq v_{s+1} + 1$ or $u_s < u_{s+2}$ we may remove the $(s + 1)$-st step.

Conversely, suppose the walk satisfies the conditions, and consider (u_s, v_s) and (u_t, v_t) for $t \geq s + 2$. We have either $v_t \geq v_s + 2$ or both $v_t = v_s + 1$ and $u_s \geq u_t$. In either case, it is illegal to make such a step directly in a minor walk, and it follows that the walk is reduced.

Now, note that a reduced walk can visit a vertex at most once, as otherwise we could remove the part of the walk from leaving a given vertex until coming back to it. For the remaining claims, consider any minor walk visiting a subset of $\{(u_1, v_1), \ldots, (u_{d+k}, v_{d+k})\}$. Such a walk must begin with (u_1, v_1) and end with (u_{d+k}, v_{d+k}), so for each s, it must at some point pass from the set $\{(u_1, v_1), \ldots, (u_s, v_s)\}$ to $\{(u_{s+1}, v_{s+1}, \ldots, (u_{d+k}, v_{d+k})\}$. By the reasoning of the last paragraph, the only legal step accomplishing this is $(u_s, v_s), (u_{s+1}, v_{s+1})$. This establishes the remaining claims. □

Let G be the set of minors corresponding to reduced walks. From the second claim of Lemma 11, every leading term not divisible by another is represented in G exactly once, and no others are. From the definition of reduced, it is clear that the leading term of any minor is divisible by that of one in G. Furthermore, by the same Lemma, it is clear that reduced walks must have length at most $2d$ and that this length is achievable when $n \geq 2$. Walks of length at most $2d$ correspond to minors of M_k, $k \leq d$, which accounts for the reason the claim of Theorem 4 is as it is and is sharp.

Now let us determine $V(\operatorname{lt} G)$, which consists of coordinate subspaces. The equations of one of the components consist of a minimal subset of variables so that all generators of $\operatorname{lt} G$ are divisible by at least one in the subset.

Proposition 12. *Let $1 \leq s \leq n$ and $1 \leq t \leq d$. Write*

$$S_{s,t} = \{a_{i,t-1}, i < s\} \cup \{a_{i,t}, i > s\}.$$

Then $S_{s,t}$ is an inclusion minimal subset of variables intersecting the vertices of every (reduced) minor walk, and all such subsets are one of the $S_{s,t}$.

Thus, there are nd such subsets, each of size $n-1$. In particular $V(\operatorname{lt} G)$ is equidimensional of projective dimension nd and degree nd.

Proof. A minor walk must intersect $S_{s,t}$ at the beginning or end of any step it passes from column $t-1$ to t, and it must do this at least once. It is easy to construct a minor walk avoiding any proper subset of $S_{s,t}$, so the first claim is shown.

Now, let S be an inclusion minimal subset intersecting any walk, and identify variables with corresponding lattice points. Suppose $(i_1, j_1), (i_2, j_2) \in S$, $i_1 \leq i_2$. Since S is minimal under inclusions, there must be a minor walk avoiding any proper subset of S, in particular there is a pair of minor walks that avoid S except for exactly (i_1, j_1) and (i_2, j_2), respectively.

Suppose either $i_1 + 2 \leq i_2$ or $i_1 + 1 = i_2$ and $j_1 \geq j_2 - 1$. Then the prefix of the path into and excluding the first occurrence of (i_2, j_2) followed by the suffix out of and excluding the last occurrence (i_1, j_1) is a minor walk avoiding S, contradiction. Hence S is confined to one column, in which case it equals $S_{1,t}$ or $S_{n,t}$, or to two adjacent columns, and any element of S in the left column rules out any in the right column above one below the element. Such a set S is then a subset of an $S_{s,t}$, and by minimality of $S_{s,t}$ equal. □

We can now finish the proof of our main result.

Proof of Theorem 4. We will prove that G is a Gröbner basis of the resultant ideal $I_{d,n}$, that is, $\operatorname{lt} G$ generates the initial ideal of $I_{d,n}$. By Proposition 12, we know that $V(\operatorname{lt} G)$ is a reduced, equidimensional variety of degree dn and dimension dn. For any ideal J that strictly contains $\operatorname{lt} G$, the variety $V(J)$ must be strictly included in $V(\operatorname{lt} G)$. In particular, it must have either strictly smaller dimension, or the same dimension and strictly smaller degree. However, $V(\operatorname{lt} I_{d,n})$ has the same dimension and degree as $V(I_{d,n})$, that is, dn and dn by Corollary 2. Thus, $\operatorname{lt} I_{d,n}$ cannot strictly contain the ideal generated by $\operatorname{lt} G$, and as $G \subset I_{d,n}$, G is a Gröbner basis for $I_{d,n}$ and in particular generates it. □

Remark 13. A question arises as to whether one could explicitly identify a *minimal* generating set for $I_{d,n}$, perhaps within our Gröbner basis G. Our work sheds no light on this interesting question. The problem of finding a small set of equations for the resultant locus, albeit in the set-theoretic sense, was studied in [7].

4. Final remarks

We have provided a complete set of generators for the resultant ideal $I_{d,n}$. We proceed to explain how this is related to equations defining $X_{d,n} = V(I_{d,n}) \subset \mathbb{P}^{n(d+1)-1}$.

Lemma 14. *If the rank of M_k is strictly smaller than $d+k$, then the rank of M_{k-1} is strictly smaller than $d+k-1$.*

Proof. Suppose for contradiction that M_{k-1} has rank $d+k-1$. We may consider M_{k-1} as an upper left submatrix of M_k. Thus the rank of M_k and M_{k-1} would have to be equal. This would be only possible if the last column of M_k is zero. But in this case, so would be the last row of M_{k-1}, which is a contradiction. □

We recover the set theoretic result by Orsinger and Kakié.

Corollary 15. *The set-theoretic zero locus of all $2d \times 2d$ minors of M_d is the variety $X_{d,n} \subset \mathbb{P}^{n(d+1)-1}$.*

As noted before, in general these minors clearly cannot generate the ideal $I_{d,n}$, as smaller minors have smaller degree. However, the following result holds[2].

Proposition 16. *Let $J_{d,n}$ be the ideal generated by all $2d \times 2d$ minors of M_d, and let $m_{d,n}$ be the irrelevant ideal in $K[a_{i,j}]$. The saturation of $J_{d,n}$ with respect to $m_{d,n}$ equals $I_{d,n}$. Equivalently, $J_{d,n}$ and $I_{d,n}$ define the same projective scheme.*

Proof. The equivalence of the two claimed statements is well known [4, Ex.II.5.10(b)]. We prove that the two projective schemes are equal. For this we need to prove that for any point in the projective space there is an affine neighbourhood on which the two schemes are equal.

First we note that the group $GL(2)$ acts on the projective space $\mathbb{P}^{n(d+1)-1}$ as the change of variables x, y. This action clearly preserves $I_{d,n}$, as we know that this is the prime ideal of the locus when the forms have a common root, and this condition does not depend on the choice of coordinates. In fact, $GL(2)$ also acts on $J_{d,n}$, which can be seen through the intrinsic description of $J_{d,n}$ as a Fitting ideal.

Pick any point $p \in \mathbb{P}^{n(d+1)-1}$ corresponding to an n-tuple of degree d polynomials. One of those polynomials must be nonzero and without loss of generality we assume it is the first one. Note that in this case, we may act with an element of $GL(2)$ so that $a_{1,0} \neq 0$. Thus to compare the projective schemes defined by $J_{d,n}$ and $I_{d,n}$ it is enough to compare them on the affine chart $a_{1,0} = 1$. Thus we have to prove that the two ideals are equal after we substitute $a_{1,0} = 1$. By definition, $J_{d,n} \subset I_{d,n}$. Pick any generator of $I_{d,n}$, that is a maximal minor of some matrix M_k. Note that M_k may be realized as a submatrix M'_k of M_d in the last $(d+k)$ columns and last nk rows. Adding rows $1, n+1, 2n+1, \dots, (d-k-1)n+1$ to those of M'_k and considering all columns, we obtain a submatrix of M_d with maximal minors equal to maximal minors of M_k after we substitute $a_{1,0} = 1$. Indeed, the chosen submatrix on first $(d-k)$ columns is upper triangular with $a_{1,0}$ on its diagonal.

This shows that the two ideals are equal after we substitute $a_{1,0} = 1$, and thus finishes the proof of the proposition. □

[2]This statement and its proof were communicated to us by Jan Stevens after we posted the first version of the article on the arXiv.

We finally note that one of the main steps of the proof was finding a square-free Gröbner basis of $I_{d,n}$. There exist other term orders that provide square-free initial ideals, which however do not choose the diagonal as the leading term.

Example 17. Consider the first non-trivial case $d = 2, n = 3$. We calculate minors and leading terms in `degrevlex` polynomial ordering using Macaulay2 [3].

```
R = QQ[a_1..a_3,b_1..b_3,c_1..c_3];
N = matrix {
{a_1,b_1,c_1,0},
{a_2,b_2,c_2,0},
{a_3,b_3,c_3,0},
{0,a_1,b_1,c_1},
{0,a_2,b_2,c_2},
{0,a_3,b_3,c_3}};

M = matrix {
{a_1,b_1,c_1},
{a_2,b_2,c_2},
{a_3,b_3,c_3}};

U = (minors(3,M) + minors(4,N))
LU = leadTerm U
```

The output is

$$a_3b_2c_1, a_3b_2b_3c_2, a_3b_1b_3c_2, a_2b_1b_3c_2, a_3b_1b_3c_1, a_2b_1b_3c_1, a_2b_1b_2c_1$$

See [1, Ex. 6.6] for a different analysis of this example.

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